

**Directed percolation with incubation times**

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We introduce a model for directed percolation with a long-range temporal diffusion, while the spatial diffusion is kept short ranged. In an interpretation of directed percolation as an epidemic process, this non-Markovian modification can be understood as *incubation times*, which are distributed accordingly to a Lévy distribution. We argue that the best approach to find the effective action for this problem is through a generalization of the Cardy-Sugar method, adding the non-Markovian features into the geometrical properties of the lattice. We formulate a field theory for this problem and renormalize it up to one loop in a perturbative expansion. We solve the various technical difficulties that the integrations possess by means of an asymptotic analysis of the divergences. We show the absence of field renormalization at one-loop order, and we argue that this would be the case to all orders in perturbation theory. Consequently, in addition to the characteristic scaling relations of directed percolation, we find a scaling relation valid for the critical exponents of this theory. In this universality class, the critical exponents vary continuously with the Lévy parameter.

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**I. INTRODUCTION**

Directed percolation (DP) [1–3] is the generic model for nonequilibrium systems which exhibit a continuous phase transition into a unique *absorbing state*. DP describes the temporal-directed spreading of a nonconserved agent in a certain medium. The agent might be bacteria in a case of epidemics in populations, fire in a burning forest, or water in a porous rock. The spreading phenomenon is characterized by two competing processes: relying on the medium conditions, the agent may multiply itself or decay at a constant rate. Depending on the balance between these two processes, the spreading may continue forever or die out after certain time. If the agent is not allowed to appear spontaneously, in the latter case the system is trapped in the absorbing state, a configuration where the stochastic fluctuations cease entirely and the system cannot escape from. These two regimes of survival and extinction are typically separated by a continuous phase transition characterized by the DP critical exponents.

The DP universality class is extremely robust, as a whole range of theoretical models seems to belong to it. Some examples include heterogeneous catalysis [4], chemical reactions [5,6], interface depinning [7,8], the onset of spatiotemporal chaos [9], flowing sand [10], and self-organized criticality [11]. The robustness of the model has led to the conjecture that two-state spreading processes with short-range interactions generically belong to the DP class, provided that quenched randomness, unconventional symmetries, and large scales due to memory effects are absent [12,13].

In the context of epidemics, DP describes infection processes without immunization and where the disease is only transmitted to nearest neighbors by direct contact. This can

be understood realizing the problem of DP on a  $(d+1)$ -dimensional lattice, where each lattice site is considered as an individual which can be *infected* (active) with probability  $p$  or *healthy* (inactive) with probability  $1-p$ . An infected individual recovers at the next time step with probability 1 and is ready to be reinfected with the same constant probability  $p$ . The susceptibility to infection is then independent of previous infections, and this ensures the absence of immunization.

Therefore, in order to make a realistic description of epidemics the effect of immunization as well as long-range interactions should be taken into account as modifications of the DP model. Immunization can be added to DP by considering a probability for subsequent infections different from the first infection probability [14]. This non-Markovian feature changes the universality class of the model to the one corresponding to dynamical percolation, also known as a general epidemic process (GEP) [15,16]. The phase diagram of this model displays a curve phase transition line connecting the GEP and DP critical points [17]. Along this line the same universality class as GEP is found. A horizontal phase transition line also separates the GEP phase from a supercritical DP behavior. The absence of scaling along this line has been shown for the case of  $1+1$  dimensions [18] and later generalized to  $d+1$  dimensions [19,20].

On the other hand, an epidemic model with long-range interactions was first suggested by Mollison [21]. This model was studied as a generalization of DP which includes spatial long-range interactions where the spreading distances follow a power-law distribution given by

$$\mathcal{P}(r)dr \sim r^{-d-\sigma}dr, \quad (1)$$

where  $d$  is the spatial dimension and  $\sigma$  is a control parameter. Asymptotically, as  $r \rightarrow \infty$ , Eq. (1) is equivalent to a Lévy distribution and  $\sigma$  is called the Lévy exponent. In this sense we can say that the particles perform Lévy flights [22]. The claim that the critical exponents should vary continuously with  $\sigma$  [23] was confirmed by theoretical renormalization-

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group analysis [24] as well as by extensive numerical simulations [25,26]. More recently these results have been generalized to branching-annihilating Lévy flights [27] and to the pair annihilation reaction  $A+A\rightarrow 0$  [28]. So far all these studies have assumed dynamic processes which are local in time. In the case of epidemics, it is assumed that the infection happens instantaneously in time.

In order to make one step forward in making a more realistic model for epidemics, in this paper we set out to study a non-Markovian modification of the DP problem which includes waiting times, or *incubation times*, between the infection and actual outbreak of the disease in a population. We assume that these incubation times  $\tau$  are distributed asymptotically as  $\tau\rightarrow\infty$ :

$$\mathcal{F}(\tau)d\tau \propto \frac{1}{\tau^{1+\kappa}}d\tau. \quad (2)$$

Here the Lévy parameter  $\kappa>0$  is a free parameter that controls the characteristic shape of the distribution. For  $\tau\rightarrow 0$  we assume  $\mathcal{F}(\tau)$  is a smooth function of  $\tau$ . For simplicity we also assume that the dynamic processes are local in space, which means that the infection can only spread by contact with nearest neighbors.

The characterization of the universality class of this problem has remained an open problem in the literature since it was first suggested in a previous work [26] in 1999. This is mainly because of the technical difficulties that arise in the field-theoretical description when long-range waiting times are introduced. In the present work, we derive a field theory for this problem and calculate the critical exponents by means of systematic perturbation theory and the  $\epsilon$  expansion. This paper is organized as follows. In Sec. II we propose a convenient approach to derive the field-theoretical action, through a generalization of the method introduced by Cardy and Sugar in [29]. We dedicate Sec. III to study and analyze the mean-field predictions of our theory. Subsequently in Sec. IV fluctuation effects are taken into account via renormalization-group methods. The various difficulties that emerge through the renormalization process are managed by studying the asymptotic behavior of the integrals involved in the renormalization process. Finally, in Sec. V we write the renormalization-group equations and compute the critical exponents at one loop. In Sec. VI, devoted to the conclusion, we argue that our results are valid to any loop order.

## II. THE MODEL

### A. Master equation

In order to derive the field-theoretical action for the problem of DP with incubation times, we first consider the master equation formalism. Directed percolation can be interpreted as a reaction-diffusion process of identical particles in a  $d$ -dimensional lattice, where multiple occupation is allowed. We call  $P(\alpha, t)$  the probability that the system will be at a given microstate  $\alpha$ . The dynamics of such a system usually is described by a master equation governing the temporal evolution of the probability distribution  $P(\alpha, t)$ , which its general form is given by

$$\frac{dP(\alpha, t)}{dt} = \sum_{\beta} R_{\beta\rightarrow\alpha} P(\beta, t) - \sum_{\beta} R_{\alpha\rightarrow\beta} P(\alpha, t). \quad (3)$$

The system goes from the microstate  $\alpha$  to the microstate  $\beta$  with a constant transition rate  $R_{\alpha\rightarrow\beta}$ . A naive generalization of this to processes involving transitions with incubation times would be

$$\begin{aligned} \frac{dP(\alpha, t)}{dt} = & \sum_{\beta} \int_{t' < t} dt' R_{\beta\rightarrow\alpha}(t-t') P(\beta, t') \\ & - \sum_{\beta} \int_{t < t''} dt'' R_{\alpha\rightarrow\beta}(t''-t) P(\alpha, t), \end{aligned} \quad (4)$$

where the transition rates  $R_{\alpha\rightarrow\beta}$  are time-dependent functions. But Eq. (4) is wrong, as the probabilities  $P(\beta, t')$  do not refer to mutually exclusive events for different times  $t'$ . Indeed, Eq. (4) does not conserve the total probability  $\sum_{\alpha} P(\alpha, t)$ . In fact Eq. (4) describes the dynamics of particles, which disappear from the lattice at time  $t'$ , until they reappear at time  $t > t'$ . This does not correspond to the dynamics with incubation times that we are trying to model.

To write a correct master equation one should add to the right-hand side of Eq. (4) an infinite number of terms which will take into account the non-disjoint nature of the events, so the master equation is replaced by an infinite hierarchy of coupled equations for multitime joint probabilities.

In order to avoid dealing with the difficulties inherent to a master equation formalism, we propose to adopt an alternative way to find the field-theoretical action. We will generalize a method first introduced by Cardy and Sugar in order to show that directed bond percolation was in the same universality class of Reggeon field theory [29].

### B. Action of DP with incubation times

In order to provide a model for epidemics with long incubation times, we consider our system on a  $(d+1)$ -dimensional lattice. We represent the spreading of the infection on the lattice through vectors (see Fig. 1). An infection vector between a lattice site  $(x, t)$  and another site  $(x', t')$  is present with probability  $p(x'-x, t'-t)$ , with  $t < t'$ . The temporal coordinate  $t$  indicates the preferred direction, and therefore the orientation of the infection vectors is always in the direction of the increasing time. The vectors can only connect nearest neighbors in space, but their range in time depends on the incubation times distributed as Eq. (2). Considering this model, the problem of epidemics with long incubation times can be interpreted as a temporal long-range directed percolation problem. We propose now to write a field theory for this model, through a generalization of the Cardy-Sugar method.

We define the connectivity function  $G(x_2, t_2; x_1, t_1)$  as the probability for two given lattice sites  $(x_1, t_1)$  and  $(x_2, t_2)$ , with  $t_1 < t_2$ , to be connected irrespective of the other sites. We say that two sites are connected if there is an infection vector present between them. Following Cardy and Sugar [29],  $G$  can be written as

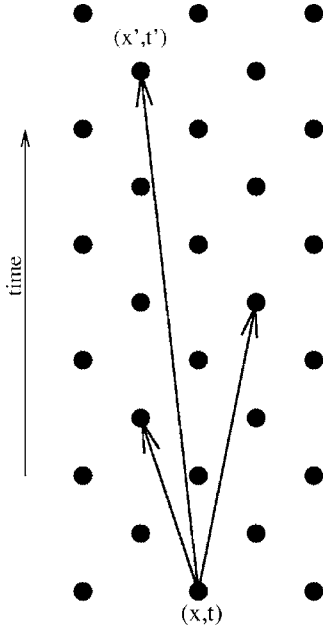


FIG. 1. Example on a (1+1)-dimensional lattice of possible temporal long-range infection vectors from a lattice site  $(x,t)$  to nearest-neighbor sites in space.

$$G(x_2, t_2; x_1, t_1) = \text{Tra}(x_2, t_2) \prod_{\text{links}, t' > t} [1 + p(x' - x, t' - t) \bar{a}(x', t') a(x, t)] \bar{a}(x_1, t_1). \quad (5)$$

The commuting operators  $a(x, t)$  and  $\bar{a}(x, t)$  act on each site  $(x, t)$  of the lattice, and their algebra is defined as [29]

$$a^2 = ia, \quad \bar{a}^2 = i\bar{a}, \quad (6)$$

$$\text{Tra}(x, t) = \text{Tr}\bar{a}(x, t) = 0, \quad (7)$$

$$\text{Tr}[a(x, t)\bar{a}(x, t)] = 1. \quad (8)$$

We remark that Eq. (5) is identical to the one obtained by Cardy and Sugar in [29], except for the fact that in our case the probability  $p$  is not a constant.

The physical features of the problem and the details of the dynamics are included in the effective lattice determined by the infection vectors. We define a matrix  $V$ , which will contain this information as follows:

$$\prod_{\text{links}, t' > t} [1 + p(x' - x, t' - t) \bar{a}(x', t') a(x, t)] = \exp\left(\sum_{x,t} \sum_{x',t'} \bar{a}(x', t') V(x' - x, t' - t) a(x, t)\right). \quad (9)$$

Therefore the matrix  $V$  will contain the information of the temporal long-range processes. In order to complete the generalization of the Cardy-Sugar method for the problem of DP with incubation times, we assume that  $V$  can be decomposed into a short-range part  $V_s(x, t)$  and other part  $V_l(x, t)$  which will be long range in time,

$$V(x, t) = V_s(x, t) + V_l(x, t). \quad (10)$$

$V_l(x, t)$  contains the long-range temporal dependence and a factor with a spatial dependence, which is short ranged. Therefore, we can assume that the leading behavior of  $V_l(x, t)$  is as follows:

$$V_l(x, t) \sim \frac{1}{t^{1+\kappa}}, \quad (11)$$

with a proportionality factor that is some short-range function of  $x$ . We consider an expansion of the Fourier-Laplace transform of  $V_s$  in a small momentum  $k$  and a small energy  $E$ ,

$$\tilde{V}_s(k, E) = \sum_{x,t} \left(1 - \frac{1}{2}(kx)^2 - Et + \dots\right) V_s(x, t), \quad (12)$$

where

$$\tilde{V}_s(k, E) = c - c_1 E - c_2 k^2 + \dots \quad (13)$$

In the case of long-range temporal processes considered here, the moment  $\langle t \rangle$  is divergent and we cannot perform an expansion in  $E$ . Instead, we compute the Laplace transform of  $\mathcal{F}(t)$  as

$$\int_0^\infty e^{-Et} \mathcal{F}(t) dt \sim E^\kappa + \text{const} + \text{regular terms}. \quad (14)$$

Therefore, the Fourier-Laplace transform of the long-range contribution  $V_l(x, t)$  will involve a  $E^\kappa$  dependence multiplied by the Fourier expansion of a spatial short-range factor,

$$\tilde{V}_l(k, E) \sim (E^\kappa + \text{const} + \dots)(1 - bk^2 + \dots). \quad (15)$$

We should notice that this is valid for values of  $\kappa > 0$ . Keeping the most relevant terms in a small- $k$  and  $-E$  expansion, the Fourier-Laplace transform of  $V$  is given by

$$\tilde{V}(k, E) = c[1 - r_1 E - r_2 k^2 - r E^\kappa + O(k^2 E^\kappa)]. \quad (16)$$

Applying Gaussian integrations in Eq. (9), we can be written as

$$\begin{aligned} & \exp\left(\sum_{x,t} \sum_{x',t'} \bar{a}(x', t') V(x' - x, t' - t) a(x, t)\right) \\ &= \int_{-\infty}^\infty \prod_{(x,t)} d\phi d\bar{\phi} \exp[\bar{\phi}(x', t')(V)^{-1} \phi(x, t) - a\bar{\phi} - \bar{a}\phi], \end{aligned} \quad (17)$$

where the operator  $V^{-1}$  is given by

$$V^{-1} = c^{-1}[1 + r_1 \partial_t - r_2 \nabla^2 + r \partial_t^\kappa + \dots]. \quad (18)$$

Finally using Eqs. (9) and (17), and replacing the result in Eq. (5) we obtain

$$G = \text{Tra}(x_2, t_2) \bar{a}(x_1, t_1) \int D\phi D\bar{\phi} \exp\left(-\int dt d^d x [\bar{\phi}(\lambda V)^{-1} \phi - a\bar{\phi} - \bar{a}\phi]\right). \quad (19)$$

Performing the trace operation in Eq. (19) and after applying a rescaling of the fields, we find the effective action of the problem of directed percolation with incubation times,

$$S = \int dt d^d x \left[ \bar{\phi}(\partial_t^\kappa + \tau\partial_t - D_0\nabla^2 + r_0)\phi + \frac{1}{2}u_0\bar{\phi}\phi^2 - \frac{1}{2}u_0\bar{\phi}^2\phi \right]. \quad (20)$$

Here  $r_0 \propto p_c - p$  and the fields  $\phi$  and  $\bar{\phi}$  depend on space and time. For  $\kappa=1$ , Eq. (20) is reduced to the DP case. Thus, in our problem we consider  $0 < \kappa < 1$ . The generalization of the Cardy-Sugar formalism, in contrast with the master equation method, does not simply add the rates for any possible infection from time  $t$  to another time  $t'$  in the future, but it properly takes into account the multiple counting of events which may occur if there is more than one way of reaching a given state at time  $t'$ .

### III. MEAN-FIELD APPROXIMATION

#### A. Critical exponent $\beta$

In this section we will find the mean-field values of the critical exponents of the theory. If fluctuations effects are neglected, the field  $\phi$  can be interpreted as a density field, and consequently above criticality it scales as

$$\phi \propto |p_c - p|^\beta, \quad p > p_c. \quad (21)$$

We start by finding the classical equations of motion. In order to do so we consider a variation of the action in Eq. (20), with respect to the fields  $\phi$  and  $\bar{\phi}$ . If we define the Lagrangian density as

$$\mathcal{L} = \bar{\phi}(\partial_t^\kappa + \tau\partial_t - D_0\nabla^2 + r_0)\phi + \frac{1}{2}u_0\bar{\phi}\phi^2 - \frac{1}{2}u_0\bar{\phi}^2\phi, \quad (22)$$

the corresponding equations of motion are

$$\partial_t^\kappa\phi + r_0\phi + \frac{1}{2}u_0(\phi^2 - 2\bar{\phi}\phi) - D_0\nabla^2\phi = 0, \quad (23)$$

$$r_0\bar{\phi} + \frac{1}{2}u_0(2\bar{\phi}\phi - \bar{\phi}^2) + (-1)^\kappa\partial_t^\kappa\bar{\phi} - D_0\nabla^2\bar{\phi} = 0, \quad (24)$$

after using Eq. (22). A solution where  $\bar{\phi}=0$  would be equivalent to not considering the noise fluctuations in the Langevin-like equation. From Eq. (24) we see that  $\bar{\phi}=0$  is indeed a classical solution, as long as Eq. (23) is verified:

$$D_0\nabla^2\phi - \partial_t^\kappa\phi = r_0\phi + \frac{1}{2}u_0\phi^2. \quad (25)$$

A particular solution of this equation of motion can be obtained if we neglect the temporal and spatial dependence of the field  $\phi$ —that is, a mean-field approximation. Therefore, Eq. (25) becomes

$$r_0\phi + \frac{1}{2}u_0\phi^2 = 0, \quad (26)$$

giving two solutions, for  $r_0 > 0$  (below criticality),

$$\phi = 0, \quad p < p_c, \quad (27)$$

and for  $r_0 < 0$  (above criticality),

$$\phi = \frac{-2r_0}{u_0}, \quad p > p_c. \quad (28)$$

Therefore, in a mean-field approximation  $\phi \sim r_0$ , and from here we obtain that

$$\beta^{MF} = 1. \quad (29)$$

#### B. Critical exponents $\nu_\perp$ and $\nu_\parallel$

In order to calculate the exponents  $\nu_\perp$  and  $\nu_\parallel$ , we will analyze the scaling behavior of the correlation function  $G^{(1,1)}(x, t)$ , around the Gaussian fixed point when the interaction terms in the action are neglected. Below criticality, we do not expect any longer a temporal exponential decay of  $G^{(1,1)}(x, t)$ , as happens in the case of pure DP, but a power-law behavior. This is due to the fact that the infections can happen at very large times. However, the spatial decay of  $G^{(1,1)}(x, t)$  is an exponential decay, since the spatial diffusion is short ranged. Therefore  $G^{(1,1)}(x, t)$  decays exponentially in the limit of large  $x$  and as a power law in the limit of large times. We proceed to write the Fourier-Laplace transform of  $G^{(1,1)}(x, t)$  as

$$G^{(1,1)}(x, t) = \int \frac{d^d k}{(2\pi)^d} e^{ikx} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dE}{2\pi i} \frac{e^{Et}}{E^\kappa + D_0 k^2 + r_0}. \quad (30)$$

If we make the change of variables  $E = E' r_0^{1/\kappa}$  and  $k = \frac{k'}{D_0^{1/2} r_0^{1/2}}$ , the correlation function can be rewritten as follows:

$$G^{(1,1)}(x, t) = r_0^{1/\kappa} \left( \frac{r_0^{1/2}}{D_0^{1/2}} \right)^d F \left( r_0^{1/\kappa} t, \frac{r_0^{1/2} x}{D_0^{1/2}} \right). \quad (31)$$

Consequently, time and space scale as  $t \sim r_0^{-1/\kappa}$  and  $x \sim r_0^{-1/2}$ , respectively. At criticality  $r_0=0$ , and therefore any temporal and spatial scale is divergent. We can then define the critical exponents  $\nu_\perp$ , which describes how the spatial correlation length diverges at criticality, the exponent  $\nu_\parallel$ , describing the divergent behavior of the temporal correlation length, and the dynamic exponent  $z = \nu_\parallel / \nu_\perp$ , such that

$$\nu_{\perp}^{MF} = \frac{1}{2}, \quad \nu_{\parallel}^{MF} = \frac{1}{\kappa}, \quad z^{MF} = \frac{2}{\kappa}. \quad (32)$$

The value of these exponents are given at a mean-field level, as we have derived them neglecting the interactions in the action in order to compute  $G^{(1,1)}(x, t)$ .

Next, we will find how  $G^{(1,1)}(x, t)$  decays below criticality. We should notice that the Laplace transform involved in Eq. (30) cannot be solved exactly. Thus, in Appendix A we compute how this integral behaves asymptotically in the limit of  $t \rightarrow \infty$ , finding

$$G^{(1,1)}(x, t) \sim \frac{1}{t^{1+\kappa}} \quad (p < p_c). \quad (33)$$

We should compare this result with the DP problem, where  $G^{(1,1)}(x, t)$  decays exponentially below  $p_c$ . At criticality we can perform similar calculations setting  $x \rightarrow \infty$  (see Appendix A), and obtain

$$G^{(1,1)}(\infty, t) \sim \frac{1}{t^{1-\kappa}} \quad (p = p_c). \quad (34)$$

Consequently we find that  $G^{(1,1)}(x, t)$  behaves with different power laws at criticality and below criticality. At criticality,  $G^{(1,1)}$  follows a power-law decay given by an exponent:

$$\delta^{MF} = 1 - \kappa. \quad (35)$$

In this way we have shown that at a mean-field level, the critical behavior can be described by continuously varying exponents with the Lévy parameter  $\kappa$ . For  $\kappa=1$ , we recover the DP exponents.

#### IV. FIELD-THEORETICAL ANALYSIS

In this section we will include the effect of fluctuations in our analysis, and therefore a mean-field approach cannot be considered any longer. Instead we will apply field-theoretical techniques which will allow us to perform the calculation of the critical exponents below the upper critical dimension. We start by computing the canonical dimensions for the various quantities appearing in the action in Eq. (20), simply by considering the dimensionless nature of the action. In addition, the time-reversal symmetry ( $\tilde{\phi} \rightarrow -\phi; \phi \rightarrow -\tilde{\phi}$ ), still valid in this problem, suggests the use of equal canonical dimensions for both fields,  $\tilde{\phi}$  and  $\phi$ . Therefore,

$$[\tilde{\phi}] = [\phi] = \omega^{(1-\kappa)/2} k^{d/2}, \quad (36)$$

and the dimensions of the fields depend on the Lévy parameter  $\kappa$ . The canonical dimensions of the diffusion constant  $D_0$  and the coupling constant  $u_0$  are

$$[D_0] = \omega^{\kappa} k^{-2} \quad (37)$$

and

$$[u_0] = \omega^{(3\kappa-1)/2} k^{-d/2}, \quad (38)$$

respectively. In order to calculate the upper critical dimension  $d_c$ , we express the canonical dimension of the coupling constant in terms of momentum units only, as follows:

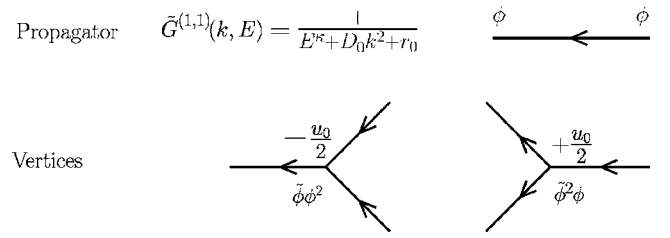


FIG. 2. Propagator and vertices for the problem of directed percolation with incubation times.

$$\left[ \frac{u_0^2}{D_0^{(3\kappa-1)/\kappa}} \right] = k^{(6\kappa-2)/\kappa-d}. \quad (39)$$

Hence, we see from Eq. (39) that the coupling constant becomes dimensionless at the value of the upper critical dimension  $d_c$ ,

$$d_c = \frac{6\kappa-2}{\kappa}, \quad (40)$$

below which the fluctuation effects become important. We should notice that Eq. (40) gives a negative  $d_c$  when  $\kappa \leq 1/3$ . For these values of  $\kappa$ , a mean-field theory rather than a field-theoretical approach should be implemented. Consequently, in this section we only consider  $1/3 < \kappa < 1$ .

The Feynman rules for the propagator and the vertices of the theory are formulated in Fig. 2. The propagator  $G^{(1,1)} \times(k, E)$  is represented by a straight line, and its expression can be obtained from the free action, taking the Laplace transform of time and the Fourier transform of the spatial dimensions into momentum space, given as

$$\tilde{G}^{(1,1)}(k, E) = \frac{1}{E^\kappa + D_0 k^2 + r_0}. \quad (41)$$

We have neglected in Eq. (41) the linear term  $\tau E$  with respect to  $E^\kappa$ , since in the low-energy limit ( $E \rightarrow 0$ ), the latter term is dominant. The main difference with respect to DP is the modification of the propagator due to the long-range temporal infections, given by the non-Markovian term  $E^\kappa$ . Notice that the vertex interactions are not altered with respect to DP.

In what remains of this section we proceed with the renormalization of the theory. We will apply mass, field, and diffusion constant renormalizations to absorb the divergences of the two-point vertex function  $\Gamma^{(1,1)}$ . The divergences of  $\Gamma^{(2,1)}$  will be considered in the coupling constant renormalization, and finally we will renormalize the composite operator ( $\tilde{\phi}\phi$ ).

##### A. Mass and field renormalizations

We start by writing the two-point vertex function  $\Gamma^{(1,1)}$  at one loop. Figure 3 shows the diagrammatic expansion up to one loop—that is,

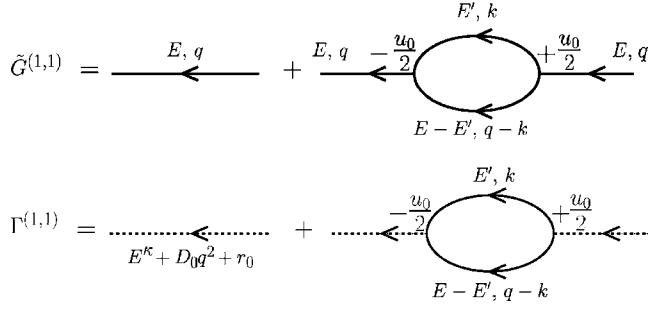


FIG. 3. Feynman diagrams which contribute to the expansions of the propagator  $\tilde{G}^{(1,1)}$  and the two-point vertex function  $\Gamma^{(1,1)}$  up to one loop.

$$\Gamma^{(1,1)} = E^\kappa + D_0 q^2 + r_0 + \frac{u_0^2}{2} \int \frac{dE'}{(2\pi i)} \int \frac{d^d k}{(2\pi)^d} \times \frac{1}{[E'^\kappa + D_0 k^2 + r_0][(E-E')^\kappa + D_0(q-k)^2 + r_0]}. \quad (42)$$

This vertex function has two kinds of divergences. One kind may happen at  $d=4-2/\kappa$ , and we assume it is absorbed into a redefinition of the bare mass  $r_0$  to a renormalized mass  $r_R$ . A second kind of divergence may happen at  $d=6-2/\kappa$ , and it will be absorbed into a renormalization constant of the fields.

We will work in a Laplace-Fourier space constituted by an energy  $E$ , considered positive and real, and a momentum  $k$ . We define the normalization point (NP) such that the external energy is evaluated at an arbitrary scale  $E=\zeta$ , while the external momentum is set to zero—that is,  $q=0$ . We define the first renormalization condition

$$\left. \frac{\partial \Gamma_R^{(1,1)}}{\partial (E^\kappa)} \right|_{NP} = 1. \quad (43)$$

The renormalization of the fields defines the renormalization constants  $Z_\phi$  and  $Z_{\tilde{\phi}}$ , such that

$$\phi_R = Z_\phi^{-1/2} \phi, \quad \tilde{\phi}_R = Z_{\tilde{\phi}}^{-1/2} \tilde{\phi}. \quad (44)$$

Nevertheless, due to the time-reversal symmetry, we can choose  $Z_\phi = Z_{\tilde{\phi}}$ . The two-point vertex function, calculated by cutting off the external propagators to the correlation function  $\tilde{G}^{(1,1)}$ , is then

$$\Gamma_R^{(1,1)} = (\tilde{G}^{(1,1)})^{-1} = Z_\phi \Gamma^{(1,1)}. \quad (45)$$

Inserting this into the renormalization condition, Eq. (43), we obtain the expression for the field renormalization constant,

$$Z_\phi^{-1} = \left. \frac{\partial \Gamma^{(1,1)}}{\partial (E^\kappa)} \right|_{NP}. \quad (46)$$

Unfortunately, we were unable to evaluate the integral involved in the expression of  $\Gamma^{(1,1)}$  in Eq. (42) exactly, and consequently we must rely on an analysis of its asymptotic behavior at the singular points. Applying standard complex variable theory, we can see that the integral in Eq. (42) pre-

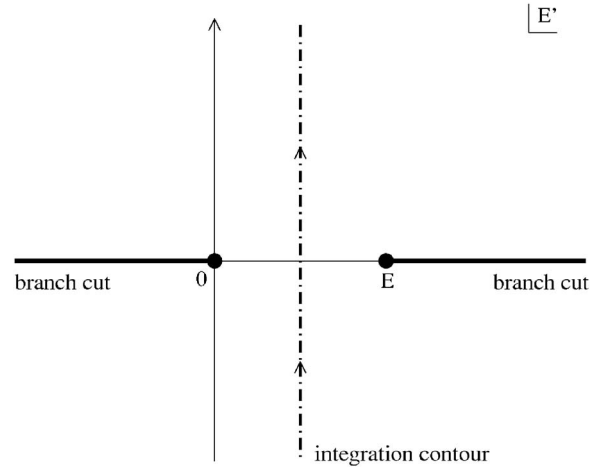


FIG. 4. Branch points and branch-cut topology for the integral in Eq. (46).

sents two logarithmic branch points: if we write  $E'^\kappa = e^{\kappa \ln E'}$ , we identify one branch point in  $E'=0$ . In the same way it is possible to see that  $E'=E$  is the second branch point. There are no poles in the first Riemann sheet, and therefore we consider the branch-cut topology shown in Fig. 4.

We will replace the bare mass  $r_0$  by the renormalized mass  $r_R$ , assuming that the mass renormalization is already done. For simplicity, when evaluating the integrals in Eq. (46), we will set  $r_R=0$  and  $q=0$ . We start by performing the momentum integration which can be solved exactly in a straightforward manner, giving

$$\Gamma^{(1,1)}|_{NP} = E^\kappa + \frac{u_0^2}{2} S_d \frac{\pi}{2} \csc\left(\frac{d\pi}{2}\right) \frac{1}{D_0^{d/2}} E^{(1-\epsilon/2)\kappa} I \Big|_{E=\zeta}, \quad (47)$$

with

$$I = \int_{1/2-i\infty}^{1/2+i\infty} \frac{du}{(2\pi i)} \frac{u^{\kappa(d/2-1)} - (1-u)^{\kappa(d/2-1)}}{(1-u)^\kappa - u^\kappa}. \quad (48)$$

In Eq. (47),  $\epsilon = d_c - d$  and  $S_d = 2\pi^{d/2}/\Gamma(d/2)$ . We have made the change of variable  $E' = Eu$ , where  $E$  is real and positive. Taking the derivative with respect to  $E^\kappa$  and evaluating in the NP we obtain the expression for the field renormalization constant  $Z_\phi$ :

$$Z_\phi^{-1} = 1 + \frac{u_0^2}{4} S_d \csc\left(\frac{d\pi}{2}\right) \frac{\pi}{D_0^{d/2}} \left(1 - \frac{\epsilon}{2}\right) \zeta^{-\kappa\epsilon/2} I. \quad (49)$$

In order to solve the integral  $I$ , we notice that there are two branch points present, one at  $u=0$  and another at  $u=1$ . We were unable to evaluate this integral exactly, and consequently in Appendix B 1 we analyze the asymptotic behavior of the integral at the possible points where divergences may occur. We find that there is no other divergence present in Eq. (49), except for the one reabsorbed in the definition of the renormalized mass  $r_R$ . The direct consequence of this is that the field renormalization coefficient remains constant—that is,  $Z_\phi = 1 + \text{const}$ —and under a suitable rescaling of the

fields it is possible to redefine  $Z_\phi$  such that at one-loop order we have

$$Z_\phi = 1. \quad (50)$$

In conclusion, according to one-loop calculations, the field renormalization is not required in the theory and

$$\Gamma_R^{(1,1)}|_{NP} = \Gamma^{(1,1)}|_{NP}, \quad (51)$$

which proves that the bare propagator is the full propagator for our theory.

### B. Diffusion constant renormalization

In the case of the diffusion constant renormalization we proceed in a similar manner to that described in the previous subsection, since in this case we also have the technical difficulty that the integrals involved in the renormalization cannot be calculated exactly. Then, we analyze the asymptotic behavior of the integrals at the different singularities in order to determine the divergences.

We impose the renormalization condition

$$\left. \frac{\partial \Gamma_R^{(1,1)}}{\partial q^2} \right|_{NP} = D_R \equiv Z_D D_0, \quad (52)$$

which defines the renormalized diffusion constant  $D_R$  and the renormalization constant  $Z_D$ :

$$Z_D = D_0^{-1} \left. \frac{\partial \Gamma^{(1,1)}}{\partial q^2} \right|_{NP}. \quad (53)$$

We simplify our calculations setting from the beginning in our analysis  $r_R=0$ . After performing the momentum integration we obtain

$$\begin{aligned} \left. \frac{\partial \Gamma^{(1,1)}}{\partial q^2} \right|_{E=\zeta} &= D_0 + \frac{u_0^2}{2} \frac{S_d}{4(2\pi)^d D_0^{d/2-1}} \pi \csc\left(\frac{d\pi}{2}\right) \\ &\times \left[ -I_{E_1} + \frac{1}{d} I_{E_2} \right], \end{aligned} \quad (54)$$

where the integrals  $I_{E_1}$  and  $I_{E_2}$  are given in Appendix B 2 by Eqs. (B6) and (B7), respectively. In Appendix B 2 we obtain the divergences of these integrals through an analysis of their asymptotic behavior at the singular points. At the upper critical dimension, we find logarithmic divergences. The integrals cannot be solved analytically, and therefore we calculate the coefficients of the divergences. The results are given by Eqs. (B14) and (B15). Inserting these results into Eq. (54), the expression for  $D_R$  is given by

$$\begin{aligned} D_R = D_0 Z_D &= D_0 + \frac{u_0^2}{D_0^{d/2-1}} \frac{S_d}{2^{\kappa(d/2-3)+5} (2\pi)^d} \\ &\times \csc\left(\frac{d\pi}{2}\right) \frac{\zeta^{-\kappa\epsilon/2}}{\kappa\epsilon} [F_1(\kappa) + F_2(\kappa)], \quad \epsilon \rightarrow 0, \end{aligned} \quad (55)$$

where the functions  $F_1(\kappa)$  and  $F_2(\kappa)$  are

$$F_1(\kappa) = 2 \left( 2 - \frac{1}{\kappa} \right) \frac{\sin(\pi\kappa)}{\sin^2\left(\frac{\pi}{2}\kappa\right)} \quad (56)$$

and

$$F_2(\kappa) = \frac{1}{\kappa \sin^3\left(\frac{\pi}{2}\kappa\right)} \left[ \cos\left(\frac{\pi}{2}\kappa\right) + \frac{4\kappa^2 - 3\kappa + 1}{3\kappa - 1} \cos\left(\frac{3\pi}{2}\kappa\right) \right]. \quad (57)$$

Although the approximations we have used to compute the behavior of the integrals are rather drastic, they lead one to obtain the coefficients of the divergences in a transparent way.

### C. Coupling constant renormalization

Turning now to the coupling constant renormalization, we wish to construct and follow an equivalent procedure to tackle the integrations that will appear in this part of the renormalization process. We start by defining the renormalized coupling constant  $u_R$  as

$$u_R = -\Gamma_R^{(2,1)}|_{NP} = -Z_\phi^{3/2} \Gamma^{(2,1)}|_{NP} = -\Gamma^{(2,1)}|_{NP}. \quad (58)$$

We should notice that in Eq. (58) we have used the fact that there is no field renormalization and therefore substituted  $Z_\phi=1$ . In Fig. 5 we show the Feynman diagrams contributing to  $\Gamma^{(2,1)}$  up to one loop.

The dressed vertex function  $\Gamma^{(2,1)}$ , truncated at two-loop order is, then,

$$\begin{aligned} \Gamma^{(2,1)} &= -u_0 + u_0(-u_0)^2 \int \frac{dE'}{(2\pi i)} \int \frac{d^d k}{(2\pi)^d} \\ &\times \frac{1}{(E'^\kappa + D_0 k^2)[(E_1 - E')^\kappa + D_0(q_1 - k)^2]} \\ &\times \frac{1}{[(E_1 + E_2 - E')^\kappa + D_0(q_1 + q_2 - k)^2]} \\ &+ \text{permutation}(E_1, q_1 \leftrightarrow E_2, q_2). \end{aligned} \quad (59)$$

The second Feynman diagram contribution, represented by a permutation  $(E_1, q_1 \leftrightarrow E_2, q_2)$ , is an integration with value equal to the first integration in Eq. (59), but with the external energy  $E_1$  and momentum  $q_1$  exchanged with  $E_2$ , and  $q_2$ , respectively. In order to make the computation of the integral in Eq. (59) easier, we choose the renormalization point NR as  $q_1=q_2=0$ ,  $E_2=0$ , and  $E_1=\zeta$ . This choice is arbitrary and should not modify the final results. Thus, evaluating Eq. (59) in the normalization point, we have

$$\begin{aligned} \Gamma^{(2,1)} &= -u_0 + 2u_0^3 \int \frac{dE'}{(2\pi i)} \int \frac{d^d k}{(2\pi)^d} \\ &\times \frac{1}{(E'^\kappa + D_0 k^2)[(E - E')^\kappa + D_0 k^2]^2}. \end{aligned} \quad (60)$$

The factor of 2 comes from counting the contribution of the second Feynman diagram. We proceed by integrating first over the momentum, and we find

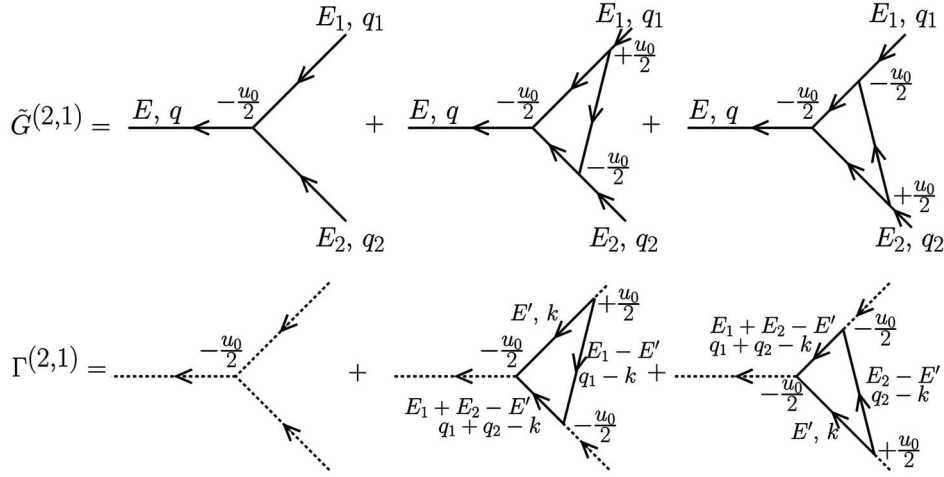


FIG. 5. Perturbative expansions for the correlation function  $\tilde{G}^{(2,1)}$  and the corresponding vertex function  $\Gamma^{(2,1)}$  up to one loop.

$$\Gamma^{(2,1)} = -u_0 \left[ 1 - \frac{u_0^2}{D_0^{d/2}} \frac{\pi \csc(d\pi/2) S_d}{(2\pi)^d} I_{E_1} \right]. \quad (61)$$

Remarkable, the  $I_{E_1}$  is the same integral which first appeared in the  $D_0$  renormalization. We then make use of the result in Eq. (B14) and inserting it into Eq. (59) we finally obtain

$$u_R = u_0 \left[ 1 + \frac{u_0^2}{D_0^{d/2}} \frac{\csc(d\pi/2) S_d}{2^{\kappa(d/2-3)+2} (2\pi)^d} F_1(\kappa) \frac{\zeta^{-\kappa\epsilon/2}}{\kappa\epsilon} \right], \quad \epsilon \rightarrow 0. \quad (62)$$

We are now able to define and calculate the dimensionless renormalized coupling constant  $g_R$ . Accordingly to Eq. (39), it can be defined as

$$g_R = \frac{u_R}{D_R^{d/4}} \mu^{-\epsilon/2}, \quad (63)$$

where  $\mu$  is a momentum scale and therefore related to  $\zeta$  by  $\mu = \zeta^{\kappa/2} / D_R^{1/2}$ . As we have used an energy scale in the renormalizations, it is convenient to write  $g_R$  in terms of  $\zeta$ , as follows:

$$g_R = \frac{u_R}{D_R^{(d-\epsilon)/4}} \zeta^{-\epsilon\kappa/4}. \quad (64)$$

Inserting the expansions of  $D_R$  [Eq. (55)] and  $u_R$  [Eq. (62)], into Eq. (64) we have finally

$$g_R = \frac{u_0}{D_0^{(d-\epsilon)/4}} \zeta^{-\epsilon\kappa/4} \left[ 1 + \left( \frac{u_0}{D_0^{d/4}} \zeta^{-\epsilon\kappa/4} \right)^2 \times \frac{S_d \csc(d\pi/2)}{2^{\kappa(d/2-3)+7} (2\pi)^d} \frac{1}{\kappa\epsilon} F_3(\kappa) \right], \quad \epsilon \rightarrow 0, \quad (65)$$

where

$$F_3(\kappa) = (32 - d_c) F_1(\kappa) + d_c F_2(\kappa). \quad (66)$$

The next subsection is dedicated to study the final renormalization step: the composite operator  $(\tilde{\phi}\phi)$  renormalization.

#### D. Composite operator renormalization

We start by defining the renormalized composite operator  $(\tilde{\phi}\phi)_R$  as follows:

$$(\tilde{\phi}\phi)_R = Z_{\tilde{\phi}\phi}^{-1} (\tilde{\phi}\phi). \quad (67)$$

Then the renormalized two-point correlation function with the insertion of the composite operator can be written as

$$\tilde{G}_R^{(1,1;1)} = \langle (\tilde{\phi}\phi)_R \tilde{\phi}_R \phi_R \rangle = Z_{\tilde{\phi}\phi}^{-1} \tilde{G}^{(1,1;1)}, \quad (68)$$

and the corresponding renormalized vertex function  $\Gamma_R^{(1,1;1)}$  is defined by cutting off the external propagator in Eq. (68):

$$\Gamma_R^{(1,1;1)} = [\tilde{G}_R^{(1,1)}]^{-1} \tilde{G}_R^{(1,1;1)} [\tilde{G}_R^{(1,1)}]^{-1} = Z_{\tilde{\phi}\phi}^{-1} \Gamma^{(1,1;1)}. \quad (69)$$

We impose the normalization condition

$$\Gamma_R^{(1,1;1)}|_{NP} \equiv 1, \quad (70)$$

and using Eq. (69) we obtain the expression for the renormalized constant  $Z_{\tilde{\phi}\phi}$ ,

$$Z_{\tilde{\phi}\phi} = \Gamma^{(1,1;1)}|_{NP}. \quad (71)$$

The next step is computing the unrenormalized vertex function  $\Gamma^{(1,1;1)}$  evaluated on the renormalization point NP, chosen as  $q=0$  and  $E=\zeta$ . The Feynman diagrams corresponding to the dress vertex function  $\Gamma^{(1,1;1)}$  up to one loop are shown in Fig. 6.

The simplest way to calculate  $\Gamma^{(1,1;1)}$  is, first, shifting above criticality where the renormalized mass  $r_R$  is different from its value at criticality  $r_{Rc}$ . We define a parameter  $\Delta_0 = r_R - r_{Rc}$  as a measure of the departure from criticality, and we take the derivative of  $\Gamma^{(1,1)}$  with respect to  $\Delta_0$ , as follows:



FIG. 6. Expansions of the correlation function  $G^{(1,1;1)}$  and  $\Gamma^{(1,1;1)}$  at one loop. The insertion of the composite operator ( $\tilde{\phi}\phi$ ) is represented by a cross.

$$\begin{aligned} \Gamma^{(1,1;1)}|_{NP} &= \left. \frac{\partial \Gamma^{(1,1)}}{\partial \Delta_0} \right|_{NP} \\ &= 1 + \frac{u_0^2}{2} \int \frac{dE'}{(2\pi i)} \frac{1}{(E-E')^\kappa - E'^\kappa} \\ &\quad \times \left[ \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{(E'^\kappa + D_0 k^2 + \Delta_0)^2} \right. \\ &\quad \left. + \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(E-E')^\kappa + D_0 k^2 + \Delta_0]^2} \right] \Bigg|_{E=\zeta}. \end{aligned} \quad (72)$$

Second, in order to evaluate  $\Gamma^{(1,1;1)}$  at criticality we take the limit  $\Delta_0 \rightarrow 0$ . After performing the momentum integrations, we have

$$\Gamma^{(1,1;1)}|_{NP} = 1 + \frac{u_0^2}{D_0^{d/2}} \frac{S_d}{8(2\pi)^d} (d-2)\pi \csc\left(\frac{d\pi}{2}\right) I_2, \quad (73)$$

where

$$I_2 = \int_{E/2-i\infty}^{E/2+i\infty} \frac{dE'}{(2\pi i)} \frac{E'^{\kappa(d/2-2)} - (E-E')^{\kappa(d/2-2)}}{(E-E')^\kappa - E'^\kappa} \Bigg|_{E=\zeta}. \quad (74)$$

In Appendix B 3 we obtain the asymptotic behavior of  $I_2$  at the logarithmic divergence [Eq. (B19)]. Substituting this result into the expression of  $\Gamma^{(1,1;1)}$  in Eq. (73), we have finally

$$\begin{aligned} Z_{\tilde{\phi}\phi} &= 1 - \frac{u_0^2}{D_0^{d/2}} \frac{S_d}{2^{\kappa(d/2-3)+3} (2\pi)^d} (d-2) \\ &\quad \times \frac{\sin\left[\frac{\pi}{2}\kappa\left(\frac{d}{2}-2\right)\right]}{\sin\left[\frac{\pi}{2}\kappa\right]} \frac{\zeta^{-\kappa\epsilon/2}}{\kappa\epsilon} \csc\left(\frac{d\pi}{2}\right), \quad \epsilon \rightarrow 0. \end{aligned} \quad (75)$$

In this way we complete the renormalization procedures required to absorb any possible divergent term in the vertex functions up to one loop. In the following section we will write down the renormalization-group equations and calculate the critical exponents.

## V. CALLAN-SYMANZIK EQUATION

Having performed all the renormalizations required for the theory, we are now able to calculate the renormalization-group equation for  $\Gamma_R^{(1,1)}$  and  $\Gamma_R^{(1,1;1)}$  at criticality and derive the critical exponents. We will derive the Callan-Symanzik equations considering a normalization scale  $\mu$  in units of momentum and make the change to  $\zeta$  through the relation  $\mu \frac{\partial}{\partial \mu} = \frac{2}{\kappa} \zeta \frac{\partial}{\partial \zeta}$ . Subsequently the  $\mu$  dependence will disappear when we express it in terms of physical quantities, such as energy and momentum. We start by using the fact that the unrenormalized  $\Gamma^{(1,1)}$  does not depend on the normalization scale introduced by the normalization point NP, having then

$$\left( \mu \frac{d}{d\mu} \right)_{u_0, D_0} [Z_\phi^{-1} \Gamma_R^{(1,1)}(\mu, D_R, g_R)] = 0, \quad (76)$$

and replacing now the total derivative with partial derivatives, we find

$$\left[ \mu \frac{\partial}{\partial \mu} - \gamma_\phi + \gamma_D D_R \frac{\partial}{\partial D_R} + \beta(g_R) \frac{\partial}{\partial g_R} \right] \Gamma_R^{(1,1)}(\mu, D_R, u_R) = 0, \quad (77)$$

where the flow equations, the beta and gamma functions, are define as follows:

$$\beta(g_R) = \mu \left( \frac{\partial g_R}{\partial \mu} \right)_{u_0, D_0},$$

$$\gamma_\phi = \mu \left( \frac{\partial \ln Z_\phi}{\partial \mu} \right)_{u_0, D_0}, \quad \gamma_D = \mu \left( \frac{\partial \ln Z_D}{\partial \mu} \right)_{u_0, D_0}. \quad (78)$$

The beta function can be calculated using Eq. (65), and it reads as

$$\begin{aligned} \beta(g_R) &= \frac{2}{\kappa} \zeta \left( \frac{\partial g_R}{\partial \zeta} \right)_{u_0, D_0} = -\frac{\epsilon}{2} \left( \frac{u_0}{D_0^{(d-\epsilon)/4}} \zeta^{-\epsilon\kappa/4} \right) \\ &\quad + \frac{3b}{2\kappa} \left( \frac{u_0}{D_0^{(d-\epsilon)/4}} \zeta^{-\epsilon\kappa/4} \right)^3 + O(u_0^5), \end{aligned} \quad (79)$$

where

$$b(\kappa) = \frac{-S_d \csc(d\pi/2)}{2^{\kappa(d/2-3)+7} (2\pi)^d} F_3(\kappa) \quad (80)$$

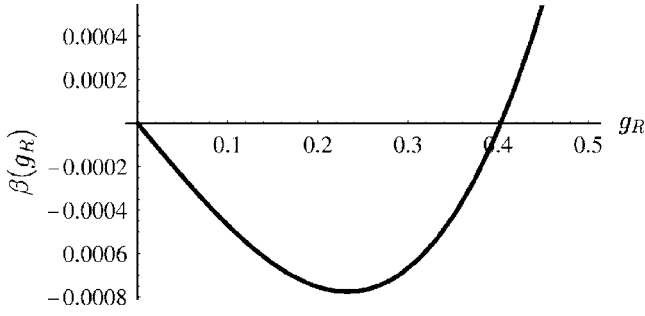
is a positive and finite function of  $\kappa$  in the domain of interest—that is,  $\frac{1}{3} < \kappa < 1$ . Up to first order in  $u_0$ ,  $g_R$  can be written as

$$g_R \sim \frac{u_0}{D_0^{(d-\epsilon)/4}} \zeta^{-\epsilon\kappa/4}, \quad (81)$$

and inserting it into Eq. (65) we obtain

$$\frac{u_0}{D_0^{(d-\epsilon)/4}} \zeta^{-\epsilon\kappa/4} = g_R + \frac{b^3}{\kappa\epsilon} g_R^3 + O(g_R^5). \quad (82)$$

Using this result we rewrite Eq. (79) obtaining an expression of the beta function in terms of  $g_R$ :

FIG. 7.  $\beta$  function  $\beta(g_R)$  for  $\kappa=0.7$  and  $\epsilon=0.01$ .

$$\beta(g_R) = -\frac{\epsilon}{2}g_R + \frac{b}{\kappa}g_R^3 + O(g_R^5), \quad (83)$$

which vanishes at the fixed point

$$g_R^* = \sqrt{\frac{\kappa\epsilon}{2b}}. \quad (84)$$

The fixed point  $g_R^*$  is an infrared-stable fixed point as is possible to see from Fig. 7. The renormalization-group equation (77), evaluated at  $g_R^*$ , can be written as

$$\left[ \mu \frac{\partial}{\partial \mu} - \gamma_\phi^* + \gamma_D^* D_R \frac{\partial}{\partial D_R} \right] \Gamma_R^{(1,1)}(\mu, D_R, u_R) = 0, \quad (85)$$

where  $\gamma_\phi^* = \gamma_\phi(g_R^*) = 0$  and  $\gamma_D^* = \gamma_D(g_R^*)$  at one loop is

$$\gamma_D^* = \frac{2}{F_3(\kappa)} [F_1(\kappa) + F_2(\kappa)] \epsilon + O(\epsilon^2). \quad (86)$$

A solution of Eq. (85) is given by

$$\Gamma_R^{(1,1)} = D_R \mu^2 \Phi \left( \frac{k}{\mu}, \frac{E}{D_R^{1/\kappa} k^{2/\kappa}} \right). \quad (87)$$

In order to write Eq. (85) in terms of the physical quantities of momentum  $k$  and energy  $E$ , we replace the derivatives on  $\mu$  and  $D_R$  using the following identities:

$$\mu \frac{\partial}{\partial \mu} = 2 - k \frac{\partial}{\partial k} - 2 \frac{1}{\kappa} E \frac{\partial}{\partial E} \quad (88)$$

and

$$D_R \frac{\partial}{\partial D_R} = 1 - \frac{1}{\kappa} E \frac{\partial}{\partial E}, \quad (89)$$

which are easy to derive from direct application of the rule of chain. In this way, we eliminate the  $\mu$  dependence in the Callan-Symanzik equation, rewriting it as

$$\left[ k \frac{\partial}{\partial k} - (2 - \gamma_\phi^* + \gamma_D^*) + \frac{1}{\kappa} (2 + \gamma_D^*) E \frac{\partial}{\partial E} \right] \Gamma_R^{(1,1)}(k, E) = 0, \quad (90)$$

and applying standard methods to solve it; we obtain

$$\Gamma_R^{(1,1)}(k, E) = k^{2 - \gamma_\phi^* + \gamma_D^*} \Phi \left( \frac{E}{k^{(2 + \gamma_D^*)/\kappa}} \right). \quad (91)$$

From this equation we can derive that  $E \sim k^{(2 + \gamma_D^*)/\kappa}$ , and considering the definition of the dynamic exponent  $z$ ,  $E \sim t^{-1} \sim k^z$ , we obtain that

$$z = \frac{2 + \gamma_D^*}{\kappa}, \quad (92)$$

which at one loop gives

$$z = \frac{2}{\kappa} + \frac{2[F_1(\kappa) + F_2(\kappa)]}{\kappa F_3(\kappa)} \epsilon + O(\epsilon^2). \quad (93)$$

This expression is valid at one-loop order in the perturbative expansion. The next step now is to determine the anomalous dimension. In order to do so, we calculate the two-point correlation function for large times  $t \rightarrow \infty$ :

$$G^{(1,1)}(x, \infty) \sim \int \frac{d^d k}{(2\pi)^d} e^{-ikx} \frac{k^{(2 + \gamma_D^*)/\kappa}}{k^{2 - \gamma_\phi^* + \gamma_D^*}} \sim k^{d - 2 + \eta_\perp}, \quad p \rightarrow p_c. \quad (94)$$

It is straightforward to find the value of the anomalous dimension through a simple comparison of the two last lines in Eq. (94), finding that  $\eta_\perp = \frac{2}{\kappa} + \gamma_\phi^* + \gamma_D^* \left( \frac{1}{\kappa} - 1 \right)$ . Using the result obtained for  $z$  in Eq. (92) and making  $\gamma_\phi^* = 0$ , we have finally

$$\eta_\perp = z(1 - \kappa) + 2. \quad (95)$$

This new relationship at one-loop order between  $\eta_\perp$  and  $z$  is a direct consequence of the absence of field renormalization. We want to compute now how the density of active particles behaves as  $x \rightarrow \infty$ —that is, the correlation function  $G_R^{(1,1)} \times (\infty, t)$ . Therefore, we turn our attention to Eq. (91), where we can see that  $k \sim E^{1/(2 + \gamma_D^*)}$ , and as  $k \rightarrow 0$ , the vertex function  $\Gamma_R^{(1,1)}$  scales as

$$\Gamma_R^{(1,1)}(0, E) \sim E^{[\kappa/(2 + \gamma_D^*)](2 - \gamma_\phi^* + \gamma_D^*)}. \quad (96)$$

It is then easy to obtain the temporal dependence of  $G_R^{(1,1)}$  at  $k=0$ , using the behavior of the vertex function given by Eq. (96):

$$G_R^{(1,1)}(x = \infty, t) \sim \int dE e^{Et} \Gamma_R^{(1,1)-1}(0, E) \sim t^{-1 + \kappa - \kappa \gamma_\phi^*/(2 + \gamma_D^*)}, \quad (97)$$

which gives a power-law decay with the exponent

$$\delta = 1 - \kappa + \frac{\kappa \gamma_\phi^*}{2 + \gamma_D^*}. \quad (98)$$

At one loop, because of the absence of field renormalization in the theory,  $\gamma_\phi^* = 0$  and therefore the value of this exponent coincides at one loop with its mean-field value found in Eq. (35). This is an expected result, in the sense that when field renormalization is not required, the bare propagator, valid to describe the density of active particles at a mean-field level, is itself the full propagator of the theory when fluctuations effects are taken into account.

**$\beta$ ,  $\nu_{\perp}$ , and  $\nu_{\parallel}$  exponents**

In this subsection we shall investigate the renormalization-group equation for the vertex function  $\Gamma^{(1,1;1)}$  at criticality. The results of these calculations will let us derive the critical exponents  $\beta$ ,  $\nu_{\perp}$ , and  $\nu_{\parallel}$ . The starting point again is the independence of the unrenormalized  $\Gamma^{(1,1;1)}$  on the momentum scale  $\mu$ , which lets us write

$$\left(\mu \frac{d}{d\mu}\right)_{u_0, D_0} Z_{\bar{\phi}\phi} Z_{\phi}^{-1} \Gamma_R^{(1,1;1)} = 0. \quad (99)$$

We should notice that the parameter which accounts for the shift of criticality  $\Delta_0 = r_R - r_{Rc}$  is taken equal zero, and we will only use it at some point in the calculations in order to do dimensional analysis. The Callan-Symanzik equation for  $\Gamma^{(1,1;1)}$  reads as follows:

$$\left(\mu \frac{\partial}{\partial \mu} + \gamma_{\bar{\phi}\phi}^* - \gamma_{\phi}^* + \gamma_D^* D_R \frac{\partial}{\partial D_R}\right) \Gamma_R^{(1,1;1)} = 0, \quad (100)$$

where we have already evaluated the gamma function  $\gamma_{\bar{\phi}\phi} = \mu \frac{\partial \ln Z_{\bar{\phi}\phi}}{\partial \mu}$  at the  $g_R^*$  fixed point. At one loop we have

$$\gamma_{\bar{\phi}\phi}^* = \frac{16(2\kappa - 1)}{\kappa F_3(\kappa) \tan\left(\frac{\pi}{2}\kappa\right)} \epsilon + O(\epsilon^2). \quad (101)$$

We will maintain explicitly  $\gamma_{\phi}^*$  in the equations and make it zero in the end, with the only purpose of pointing out the direct consequences of the absence of field renormalization at one loop. Through dimensional analysis, we infer a solution of Eq. (100) as follows:

$$\Gamma_R^{(1,1;1)} = \Phi\left(\frac{k}{\mu}, \frac{E}{D_R^{1/\kappa} k^{2/\kappa}}\right). \quad (102)$$

Making use of the identities

$$\mu \frac{\partial}{\partial \mu} = -k \frac{\partial}{\partial k} - 2 \frac{1}{\kappa} E \frac{\partial}{\partial E} \quad (103)$$

and

$$D_R \frac{\partial}{\partial D_R} = -\frac{1}{\kappa} E \frac{\partial}{\partial E}, \quad (104)$$

we can replace the derivative on  $\mu$  and  $D_R$  in terms of derivatives in momentum  $k$  and energy  $E$ , to obtain the Callan-Symanzik equation at criticality:

$$\left(k \frac{\partial}{\partial k} + \gamma_{\phi}^* - \gamma_{\bar{\phi}\phi}^* + \frac{(2 + \gamma_D^*)}{\kappa} E \frac{\partial}{\partial E}\right) \Gamma_R^{(1,1;1)} = 0. \quad (105)$$

A solution of this equation is given by

$$\Gamma_R^{(1,1;1)} = k \gamma_{\bar{\phi}\phi}^* \gamma_{\phi}^* \Phi\left(\frac{E}{k^{(2+\gamma_D^*)/\kappa}}\right). \quad (106)$$

Now, we can use the scaling form of the vertex function  $\Gamma_R^{(1,1)}$  above criticality—that is,  $\Gamma_R^{(1,1)} \sim k^{2-\gamma_{\phi}^*+\gamma_D^*} f(k^{-1/\nu_{\perp}} \Delta_0, E^{-1/\nu_{\parallel}} \Delta_0)$ —to obtain in an alternative way the scaling form of  $\Gamma_R^{(1,1;1)}$ :

$$\Gamma_R^{(1,1;1)} = \left. \frac{\partial \Gamma_R^{(1,1)}}{\partial \Delta_0} \right|_{\Delta_0=0} \sim k^{2-\gamma_{\phi}^*+\gamma_D^*-\nu_{\perp}^{-1}}. \quad (107)$$

This equation in comparison with Eq. (106) allows us to find the value of the exponent  $\nu_{\perp}$  as a function of  $\gamma_{\bar{\phi}\phi}^*$  and  $\gamma_D^*$ ,

$$\nu_{\perp} = \frac{1}{2 - \gamma_{\bar{\phi}\phi}^* + \gamma_D^*}. \quad (108)$$

At one-loop order this equation gives

$$\nu_{\perp} = \frac{1}{2} + \left(\frac{8(2\kappa - 1)}{\kappa \tan\left(\frac{\pi}{2}\kappa\right)} - F_1(\kappa) - F_2(\kappa)\right) \frac{\epsilon}{2F_3(\kappa)} + O(\epsilon^2). \quad (109)$$

In addition, through the definition of  $z$ , we know that  $\nu_{\parallel} = z\nu_{\perp}$ , and therefore we find

$$\nu_{\parallel} = \frac{1}{\kappa} \left(\frac{2 + \gamma_D^*}{2 - \gamma_{\bar{\phi}\phi}^* + \gamma_D^*}\right), \quad (110)$$

which at one loop gives

$$\nu_{\parallel} = \frac{1}{\kappa} + \frac{8(2\kappa - 1)}{\kappa^2 F_3(\kappa) \tan\left(\frac{\pi}{2}\kappa\right)} \epsilon + O(\epsilon^2). \quad (111)$$

We calculate now the  $\beta$  exponent above criticality, by writing down how  $G^{(1,1)}(x, t)$  behaves in the limit of large times:

$$\begin{aligned} G^{(1,1)}(x, \infty) &\sim \int \frac{d^d k}{(2\pi)^d} \frac{d\omega}{2\pi} e^{-ikx} k^{-2+\gamma_{\phi}^*-\gamma_D^*} g\left(\frac{k}{\Delta_0^{\nu_{\perp}}}, \frac{\omega}{\Delta_0^{\nu_{\parallel}}}\right) \\ &\sim |\Delta_0|^{\nu_{\parallel}+\nu_{\perp}(d-2+\gamma_{\phi}^*-\gamma_D^*)} \sim |\Delta_0|^{\nu_{\perp}(d+\eta_{\perp}-2)} \sim |\Delta_0|^{2\beta}. \end{aligned} \quad (112)$$

In this way, above criticality we obtain the same relationship valid for DP—that is,

$$\beta = \frac{\nu_{\perp}}{2}(d + \eta_{\perp} - 2). \quad (113)$$

We can write  $d = d_c - \epsilon$ , and thus the value of the  $\beta$  exponent at one loop can be calculated using Eq. (113), given

$$\begin{aligned} \beta &= 1 + \left(\frac{8(2\kappa - 1)}{\tan\left(\frac{\pi}{2}\kappa\right)} - [F_1(\kappa) + F_2(\kappa)]\right) \frac{3\kappa - 1}{2} \\ &\quad - \frac{\kappa F_3(\kappa)}{4} \left(\frac{\epsilon}{\kappa F_3(\kappa)} + O(\epsilon^2)\right). \end{aligned} \quad (114)$$

Nevertheless, in our theory we have in addition an extra relationship given by Eq. (95) because of the absence of field renormalization in the theory. Using both Eqs. (95) and (113), we obtain, at one-loop order,

$$2\beta = \nu_{\parallel}(1 - \kappa) + d\nu_{\perp}. \quad (115)$$

The existence of this relationship makes the exponent  $\beta$  dependent on the value of  $\nu_{\parallel}$  and  $\nu_{\perp}$ , and therefore reduces the independent critical exponents from three to two independent critical exponents, with respect to the DP theory. Therefore, we find a new scaling relation at one loop for the problem of DP with incubation times. In the following section we argue that this result is valid to any loop order in perturbation theory.

## VI. DISCUSSION

In this paper we have formulated and solved a field theory for a non-Markovian model of directed percolation with the inclusion of long-range temporal diffusion, which in a context of epidemics can be interpreted as incubation times. The incubation times are distributed following a Lévy distribution, while the spatial diffusion as well as the interactions are short ranged. We first draw the attention to the fact that the conventional approach of writing a master equation, in order to apply later a second-quantized formalism, is not a convenient way to find the field-theoretical action. This is mainly due to the fact that the master equation has an infinite number of terms, a consequence of the nondisjoint nature of events for different times. Instead, we have proposed an extension of a method introduced by Cardy and Sugar [29], where we included the details of the long-range temporal diffusion in the effective lattice determined by the infection vectors. Following this approach, in a rather simple way we have found the action of the problem.

Second, we found already at a mean-field level that the critical exponents vary continuously with the Lévy parameter, signaling the existence of a new universality class. We also found at a mean-field level that the two-point correlation function decays as a power law below criticality, instead of showing an exponential decay as in DP. This is a consequence of having infections that can be produced at very large times. We also found that this power-law decay is different from the one obtained for the two-point correlation function above criticality.

Subsequently, including fluctuation effects, we have renormalized the theory at one loop. We have calculated the renormalization-group equations and we have determined the critical exponents at one order in an  $\epsilon$  expansion. The critical exponents vary continuously with the Lévy parameter and obey at one loop an extra relationship with respect to DP, Eq. (115), which is a direct consequence of the absence of field renormalization.

We argue now that the new relationship, Eq. (115), is valid to all orders in perturbation theory, which will be true if the absence of field renormalization occurs at any loop. The absence of field renormalization just by power counting is difficult to see in the renormalization scheme applied here. This is because terms proportional to  $E^{\kappa}$  in principle can be generated at any loop. Therefore in our case it will be necessary to check that the coefficients of these terms are not divergent at the upper critical dimension. The absence of singular field renormalization to all orders is more clear in

other renormalization-group schemes—for example, the Wilson method or normalization at nonzero momentum, since in those cases the relevant Feynman amplitudes are always analytic in the external energies and momenta. Hence no terms like  $E^{\kappa}$  as  $E \rightarrow 0$  can be generated in loop diagrams, even though they are present in the bare propagator. The absence of renormalization of such singular terms in the propagator to all orders has long been known for the case of long-range ferromagnets [30] and has also been recognized for other variants of DP with long-range spatial interactions [24,26] and with both long-range spatial and temporal interactions [31].

Finally, we notice that when we set  $\kappa$  equal to 1, we do not recover the DP hyperscaling relation from Eq. (115). This is due to the fact that the validity of Eq. (115) relies on the absence of field renormalization. This does not happen in DP, where field renormalization is necessary to absorb a divergence that appears at the upper critical dimension  $d_c=4$ .

Since this work was completed, a related paper [31] has appeared. This differs from the present one in that long-range effects in both time and space are considered. The renormalization of this theory is simpler than the case considered here, because neither the coefficients of  $E^{\kappa}$  nor of  $k^{\sigma}$  in the bare propagator are renormalized, and hence there are two additional scaling relations rather than the single one found here. One of the two scaling relations obtained in [31] is equivalent to the scaling relation, Eq. (115), found in this work.

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## APPENDIX A: ASYMPTOTIC BEHAVIOR OF THE CORRELATION FUNCTION $G^{(1,1)}(x, t)$

In this appendix we compute how  $G^{(1,1)}(x, t)$ , Eq. (30), behaves asymptotically in the limit of  $t \rightarrow \infty$ . The contribution to the integral is due only to the presence of a branch point in  $E=0$ , since there are no poles in the first Riemann sheet. If we call  $y=|E|$ , then

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dE}{2\pi i} \frac{e^{Et}}{E^{\kappa} + D_0 k^2 + r_0} = \frac{1}{2\pi i} \int_0^{\infty} e^{-yt} \left[ \frac{1}{y^{\kappa} e^{-i\pi\kappa} + D_0 k^2 + r_0} - \frac{1}{y^{\kappa} e^{i\pi\kappa} + D_0 k^2 + r_0} \right] dy. \quad (A1)$$

As  $t \rightarrow \infty$ , we see from Eq. (A1) that the leading behavior of the integral comes from the integration domain for small  $y$ . We consider then a series expansion of the integrand as follows:

$$\begin{aligned} & \frac{1}{y^\kappa e^{-i\pi\kappa} + D_0 k^2 + r_0} - \frac{1}{y^\kappa e^{i\pi\kappa} + D_0 k^2 + r_0} \\ &= \frac{2i}{(D_0 k^2 + r_0)^2} \sin(\pi\kappa) y^\kappa + O(y^{2\kappa}). \end{aligned} \quad (\text{A2})$$

Inserting this result into Eq. (A1), we obtain

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dE}{2\pi i} \frac{e^{Et}}{E^\kappa + D_0 k^2 + r_0} \sim \frac{\sin(\pi\kappa)}{\pi(D_0 k^2 + r_0)^2} \frac{1}{t^{1+\kappa}} \Gamma(1+\kappa). \quad (\text{A3})$$

Therefore, the asymptotic behavior of  $G^{(1,1)}(x,t)$  as  $t \rightarrow \infty$  is given by

$$\begin{aligned} G^{(1,1)}(x,t) &\sim \frac{\Gamma(1+\kappa)\sin(\pi\kappa)}{\pi} \frac{1}{t^{1+\kappa}} \int \frac{d^d k}{(2\pi)^d} \frac{e^{ikx}}{(D_0 k^2 + r_0)^2} \\ &\sim \frac{1}{t^{1+\kappa}}, \quad t \rightarrow \infty \quad (p < p_c). \end{aligned} \quad (\text{A4})$$

A similar analysis can be carried out at criticality, where  $r_0 = 0$ . In this case we set  $k=0$ , and we have

$$\begin{aligned} G^{(1,1)}(\infty,t) &\sim \frac{1}{2\pi i} \int_0^\infty e^{-yt} \frac{2i \sin(\pi\kappa)}{y^\kappa} dy \\ &\sim \frac{\sin(\pi\kappa)}{\pi} \frac{1}{t^{1-\kappa}} \Gamma(1-\kappa), \quad t \rightarrow \infty \quad (p = p_c). \end{aligned} \quad (\text{A5})$$

## APPENDIX B: ASYMPTOTIC BEHAVIOR OF THE INTEGRALS IN THE RENORMALIZATION CALCULATIONS

### 1. Integrals involved in the field renormalization

We proceed to analyze the asymptotic behavior of integral  $I$  in Eq. (48) at the values of  $u$  where divergences may occur—that is,  $u=1/2$  and  $u=\frac{1}{2} \pm i\infty$ . Where such divergences exist, we will determine the finite coefficients of the integral. At  $u \rightarrow 1/2$ , the numerator and denominator of the integrand goes to zero. Expanding them around the point  $u=1/2$ , we find that they both go to zero with the same order in  $(u-1/2)$ . Therefore the integrand remains finite in the limit  $u \rightarrow 1/2$ , and we infer then that the integral itself does not present a divergence in the integration domain around  $u=1/2$ .

The next step is to consider the asymptotic behavior as  $u \rightarrow \frac{1}{2} \pm i\infty$ . To analyze this limit it is convenient to make a change of variables  $iy = u - \frac{1}{2}$ , and the integration  $I$  in Eq. (49) can be rewritten as follows:

$$\begin{aligned} I &= 2 \int_0^\infty \frac{dy}{2\pi} y^{\kappa(d/2-2)} \left(1 + \frac{1}{4y^2}\right)^{(d/2-2)\kappa/2} \\ &\quad \times \left[ \frac{e^{i\theta(d/2-1)\kappa} - e^{-i\theta(d/2-1)\kappa}}{e^{-i\theta\kappa} - e^{i\theta\kappa}} \right], \end{aligned} \quad (\text{B1})$$

where  $\theta \equiv \theta(y) = \arctan(2|y|)$ . We now integrate by parts in Eq. (B1) and find

$$\begin{aligned} I &= \frac{1}{\pi} \int_0^\infty dy y^{\kappa(d/2-2)} f(y) \\ &= \frac{1}{\pi} \left[ \frac{y^{\kappa(d/2-2)+1}}{\kappa(d/2-2)+1} f(y) \right]_0^\infty \\ &\quad - \int_0^\infty dy \frac{y^{\kappa(d/2-2)+1}}{\kappa(d/2-2)+1} f'(y), \end{aligned} \quad (\text{B2})$$

where  $f(y)$  is

$$f(y) = \left(1 + \frac{1}{4y^2}\right)^{(d/2-2)\kappa/2} \left[ \frac{e^{i\theta(d/2-1)\kappa} - e^{-i\theta(d/2-1)\kappa}}{e^{-i\theta\kappa} - e^{i\theta\kappa}} \right]. \quad (\text{B3})$$

The function  $f(y)$  is a finite, going to a constant as  $y \rightarrow \infty$ , and vanishes as  $y \rightarrow 0$  quickly enough for the first term on the right-hand side to be convergent for  $\kappa(d/2-2)+1 < 0$ —that is, for  $d < 4 - \frac{2}{\kappa}$ . On the other hand, for  $d \geq 4 - \frac{2}{\kappa}$ , as  $y \rightarrow \infty$ , the first term on the right-hand side diverges as  $y^{\kappa(d/2-2)+1}$ . We have assumed that such a divergence was absorbed into a renormalized mass  $r_R$  in the mass renormalization procedure: the change of variables performed in Eq. (47) scales out of the integral the energy dependence in the second term of  $\Gamma^{(1,1)}$ , and therefore the divergences corresponding to mass renormalization in  $\Gamma^{(1,1)}$  and  $Z_\phi^{-1}$  are the same. Thus, it is only the second term on the right-hand side of Eq. (B2) that may diverge at  $d = d_c = 6 - \frac{2}{\kappa}$ . There is already a pole at  $d > 4 - \frac{2}{\kappa}$  in this term, and thus in order to identify the next divergence we will extend the value of this term to  $d > 4 - \frac{2}{\kappa}$ , applying analytic continuation.

First of all, we need to determine the shape of  $f'(y)$  for large values of  $y$ . We start by considering the series expansion of  $\theta = \arctan(2y) = \frac{\pi}{2} - \frac{1}{2y} + O(y^{-3})$  valid for  $y > 1/2$ . Inserting it in the expression of  $f(y)$ , we find

$$f(y) = \frac{\sin\left[\frac{\pi}{2}\left(\frac{d}{2}-1\right)\kappa\right]}{\sin\frac{\pi}{2}\kappa} \left[ 1 + \frac{\kappa}{2} \left( \frac{1}{\tan\frac{\pi}{2}\kappa} - \frac{\left(\frac{d}{2}-1\right)}{\tan\left[\frac{\pi}{2}\left(\frac{d}{2}-1\right)\kappa\right]} \right) \frac{1}{y} + O\left(\frac{1}{y^2}\right) \right]. \quad (\text{B4})$$

It is straightforward to see from here that  $f'(y) \propto \frac{1}{y^2} + O\left(\frac{1}{y^3}\right)$ . Then, if we call  $\alpha = \kappa(d/2-2)$ , the integration in the right-hand side of Eq. (B2) behaves as

$$\int_0^{\infty} y^{\alpha+1} f'(y) dy \sim \frac{y^{\alpha}}{\alpha}, \quad y \rightarrow \infty. \quad (\text{B5})$$

This result suggests that the next pole would be at  $\alpha=0$ . Nevertheless, at the upper critical dimension  $d_c=6-2/\kappa$ ,  $\alpha$  is different from zero and takes negative values for any  $\kappa$ , with  $1/3 < \kappa < 1$ . This means that the integral in Eq. (B5) is convergent as  $y \rightarrow \infty$ . We show with this calculation that there is no other divergence present in Eq. (48).

## 2. Integrals involved in the diffusion constant renormalization

In Sec. IV B, the expression of the renormalization constant  $Z_D$  contains the integrals

$$I_{E1} = \int_{E/2-i\infty}^{E/2+i\infty} \frac{dE'}{(2\pi i)} [2E'^{\kappa(d/2-1)} + (E-E')^{\kappa(d/2-1)}(d-4) - E'^{\kappa}(E-E')^{\kappa(d/2-2)}(d-2)] [E'^{\kappa} - (E-E')^{\kappa}]^2 \Big|_{E=\zeta} \quad (\text{B6})$$

and

$$I_{E2} = \int_{E/2-i\infty}^{E/2+i\infty} \frac{dE'}{(2\pi i)} \{ [8E'^{\kappa d/2} + 2E'^{\kappa}(E-E')^{\kappa(d/2-1)}(d-4)d - E'^{2\kappa}(E-E')^{\kappa(d/2-2)}(d-2)d - (E-E')^{\kappa d/2} \times (8-6d+d^2)] [E'^{\kappa} - (E-E')^{\kappa}]^3 \} \Big|_{E=\zeta}. \quad (\text{B7})$$

Let us begin to analyze the existence of divergences in  $I_{E1}$ . We make a change of variables  $E'=Eu$ , where  $E$  is real and positive. Inserting this change into Eqs. (B6) and (B7) and after evaluating in the normalization point  $E=\zeta$ , it is possible to see that the only divergences in the integrands could appear in the limits  $u \rightarrow \frac{1}{2} \pm i\infty$ . In order to analyze these limits, we make the change of variables  $iy=u-\frac{1}{2}$  and  $I_{E1}$  and  $I_{E2}$  look as follows:

$$I_{E1} = -\frac{\zeta^{-\kappa e/2}}{4\pi} \int_0^{\infty} dy y^{\kappa(d/2-3)} \left(1 + \frac{1}{4y^2}\right)^{(\kappa/2)(d/2-3)} \left[ \frac{(d-2) \left\{ \cos \left[ \theta \kappa \left( \frac{d}{2} - 1 \right) \right] - \cos \left[ \theta \kappa \left( \frac{d}{2} - 3 \right) \right] \right\}}{\sin^2(\theta \kappa)} \right] \quad (\text{B8})$$

and

$$I_{E2} = \frac{\zeta^{-\kappa e/2}}{8\pi} \int_0^{\infty} dy y^{\kappa(d/2-3)} \left(1 + \frac{1}{4y^2}\right)^{(\kappa/2)(d/2-3)} \left\{ 2(d-4)d \sin \left[ \theta \kappa \left( \frac{d}{2} - 2 \right) \right] - (d-2)d \sin \left[ \theta \kappa \left( \frac{d}{2} - 4 \right) \right] - (16-6d + d^2) \sin \left( \theta \kappa \frac{d}{2} \right) \right\} / \sin^3(\theta \kappa), \quad (\text{B9})$$

respectively, being  $\theta \equiv \theta(y) = \arctan(2|y|)$ . Therefore as  $y \rightarrow \infty$  the integrals diverge as  $y^{\kappa(\frac{d}{2}-3)}$ , and at the upper critical dimension these divergences become logarithmic. We cannot solve the integrals analytically, and for this reason we only determine the coefficients of such divergences. We start by expressing the entire integrand as a function of  $\theta$  using  $y = \frac{\tan \theta}{2}$  and  $1 + \frac{1}{4y^2} = (\sin \theta)^{-2}$ . For instance,  $I_{E1}$  now looks

$$I_{E1} = -\frac{\zeta^{-\kappa e/2}}{4\pi} \frac{1}{2^{\kappa(d/2-3)+1}} \int_0^{\pi/2} d\theta f(\theta) \\ = -\frac{\zeta^{-\kappa e/2}}{4\pi} \frac{1}{2^{\kappa(d/2-3)+1}} \int_0^{\pi/2} d\theta \frac{1}{(\cos \theta)^{\kappa(d/2-3)+2}} \frac{(d-2) \left\{ \cos \left[ \theta \kappa \left( \frac{d}{2} - 1 \right) \right] - \cos \left[ \theta \kappa \left( \frac{d}{2} - 3 \right) \right] \right\}}{\sin^2(\theta \kappa)}. \quad (\text{B10})$$

The limit of interest is  $y \rightarrow \infty$  or, equivalently,  $\theta \rightarrow \pi/2$ . Therefore, to proceed with our analysis, we can write  $\theta = \pi/2 - \alpha$ , where  $\alpha$  is a variable which tends to zero. Substituting  $\theta$  as a function of  $\alpha$  in  $I_{E1}$  in Eq. (B10) and taking the limit  $\alpha \rightarrow 0$ , we have

$$\begin{aligned}
\int_0^{\pi/2} d\theta f(\theta) &= \int_0^{\pi/2} d\alpha \\
&\times \frac{1}{\left[ \cos\left(\frac{\pi}{2} - \alpha\right) \right]^{\kappa(d/2-3)+2}} \frac{(d-2) \left\{ \cos\left[ \left(\frac{\pi}{2} - \alpha\right) \kappa \left(\frac{d}{2} - 1\right) \right] - \cos\left[ \left(\frac{\pi}{2} - \alpha\right) \kappa \left(\frac{d}{2} - 3\right) \right] \right\}}{\sin^2\left[ \left(\frac{\pi}{2} - \alpha\right) \kappa \right]} \\
&\xrightarrow{\alpha \rightarrow 0} \frac{(d-2) \left\{ \cos\left[ \frac{\pi}{2} \kappa \left(\frac{d}{2} - 1\right) \right] - \cos\left[ \frac{\pi}{2} \kappa \left(\frac{d}{2} - 3\right) \right] \right\}}{\sin^2\left(\frac{\pi}{2} \kappa\right)} \int_0^\Delta d\alpha \frac{1}{\alpha^{\kappa(d/2-3)+2}} + \text{finite}, \quad (\text{B11})
\end{aligned}$$

where  $0 < \Delta < \pi/2$ . We can write  $\kappa\left(\frac{d}{2}-3\right)+2=1-\epsilon/2$  with  $\epsilon=d_c-d$ , and then we see that as  $\alpha \rightarrow 0$ , the integrand  $f$  in function of  $\alpha$  diverges as

$$f(\alpha) \sim a\alpha^{\kappa\epsilon/2-1}, \quad \alpha \rightarrow 0, \quad (\text{B12})$$

with

$$a = \frac{(d-2) \left\{ \cos\left[ \frac{\pi}{2} \kappa \left(\frac{d}{2} - 1\right) \right] - \cos\left[ \frac{\pi}{2} \kappa \left(\frac{d}{2} - 3\right) \right] \right\}}{\sin^2\left(\frac{\pi}{2} \kappa\right)}$$

a constant number. If we now add and subtract this divergence in the expression of  $f(\alpha)$ , we can obtain the coefficient of the logarithmic divergence as follows:

$$\int_0^{\pi/2} d\alpha f(\alpha) = \int_0^{\pi/2} d\alpha [f(\alpha) - a\alpha^{\kappa\epsilon/2-1}] + a \int_0^{\pi/2} d\alpha \alpha^{\kappa\epsilon/2-1} \sim a \frac{2}{\kappa\epsilon}, \quad \epsilon \rightarrow 0. \quad (\text{B13})$$

Inserting this result into Eq. (B10), we then have

$$I_{E1} \sim \frac{-1}{\pi 2^{\kappa(d/2-3)+2}} \frac{\zeta^{-\kappa\epsilon/2}}{\kappa\epsilon} \left[ \frac{(d-2) \left\{ \cos\left[ \frac{\pi}{2} \kappa \left(\frac{d}{2} - 1\right) \right] - \cos\left[ \frac{\pi}{2} \kappa \left(\frac{d}{2} - 3\right) \right] \right\}}{\sin^2\left(\frac{\pi}{2} \kappa\right)} \right], \quad \epsilon \rightarrow 0. \quad (\text{B14})$$

Preceding in the same way for  $I_{E2}$  we have

$$\begin{aligned}
I_{E2} \sim & \frac{1}{\pi 2^{\kappa(d/2-3)+3}} \frac{\zeta^{-\kappa\epsilon/2}}{\kappa\epsilon} \left[ 2(d-4)d \sin\left[ \frac{\pi}{2} \kappa \left(\frac{d}{2} - 2\right) \right] - (d-2)d \sin\left[ \frac{\pi}{2} \kappa \left(\frac{d}{2} - 4\right) \right] - (16-6d \right. \\
& \left. + d^2) \sin\left(\kappa \frac{d\pi}{4}\right) \right] / \sin^3\left(\frac{\pi}{2} \kappa\right), \quad \epsilon \rightarrow 0. \quad (\text{B15})
\end{aligned}$$

### 3. Integrals involved in the composite operator renormalization

The integration in  $E'$  in Eq. (74) can be studied starting by doing the change of variable  $E' = Eu$ , as we did previously, considering  $E$  a real and positive number:

$$I_2 = \zeta^{\kappa(d/2-3)+1} \int_{1/2-i\infty}^{1/2+i\infty} \frac{du}{(2\pi i)} \frac{u^{\kappa(d/2-2)} - (1-u)^{\kappa(d/2-2)}}{(1-u)^\kappa - u^\kappa}. \quad (\text{B16})$$

The integrand is not divergent as  $u \rightarrow \frac{1}{2}$ , but goes to a constant as one can see from a Taylor expansion of the integrand. We can infer then that the possible divergences may have their origin in the limits  $u \rightarrow \frac{1}{2} \pm i\infty$ . For the purpose of studying the behavior of the integral in those limits, we rewrite the integration in Eq. (B16), making the change of variables  $iy = u - 1/2$ , as follows:

$$\int_{1/2-i\infty}^{1/2+i\infty} \frac{du}{(2\pi i)} \frac{u^{\kappa(d/2-2)} - (1-u)^{\kappa(d/2-2)}}{(1-u)^\kappa - u^\kappa} = 2 \int_0^\infty \frac{dy}{2\pi} y^{\kappa(d/2-3)} \left(1 + \frac{1}{4y^2}\right)^{(d/2-3)\kappa/2} \left[ \frac{e^{i\theta(d/2-2)\kappa} - e^{-i\theta(d/2-2)\kappa}}{e^{-i\theta\kappa} - e^{i\theta\kappa}} \right]. \quad (\text{B17})$$

In the limit  $y \rightarrow \infty$ , the integrand diverges as  $y^{\kappa(d/2-3)}$ , and at the upper critical dimension this integral become logarithmic divergent. We analyze then this limit by making use of the relations  $y = \frac{\tan \theta}{2}$  and setting  $\theta = \pi/2 - \alpha$ , such that  $\alpha \rightarrow 0$ . We express Eq. (B17) as a function of  $\alpha$ :

$$2 \int_0^\infty \frac{dy}{2\pi} y^{\kappa(d/2-3)} \left(1 + \frac{1}{4y^2}\right)^{(d/2-3)\kappa/2} \times \left[ \frac{e^{i\theta(d/2-2)\kappa} - e^{-i\theta(d/2-2)\kappa}}{e^{-i\theta\kappa} - e^{i\theta\kappa}} \right] \xrightarrow{\alpha \rightarrow 0} \frac{-1}{\pi 2^{\kappa(d/2-3)+1}} \frac{\sin \left[ \frac{\pi}{2} \kappa \left( \frac{d}{2} - 2 \right) \right]}{\sin \left[ \frac{\pi}{2} \kappa \right]} \int_0^\Delta d\alpha \frac{1}{\alpha^{1-\kappa\epsilon/2}} + \text{finite}. \quad (\text{B18})$$

The integrand diverges as  $\alpha^{\kappa\epsilon/2-1}$  in the limits of  $\alpha \rightarrow 0$  and  $\epsilon \rightarrow 0$ . In order to identify the coefficient of this divergence, we add and subtract the divergence itself from Eq. (B17), finding

$$\int_{1/2-i\infty}^{1/2+i\infty} \frac{du}{(2\pi i)} \frac{u^{\kappa(d/2-2)} - (1-u)^{\kappa(d/2-2)}}{(1-u)^\kappa - u^\kappa} \sim a \frac{2}{\kappa\epsilon} + \text{finite}, \quad \epsilon \rightarrow 0, \quad (\text{B19})$$

where  $a$  is a constant equal to

$$a = \frac{-1}{\pi 2^{\kappa(d/2-3)+1}} \frac{\sin \left[ \frac{\pi}{2} \kappa \left( \frac{d}{2} - 2 \right) \right]}{\sin \left[ \frac{\pi}{2} \kappa \right]}.$$

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